Duality Theory

Primal : Min $\vec{c}\vec{x}$

subject to $A\vec{x} = \vec{b}$

$x \geq 0$

Dual : Maximize $\vec{w}\vec{b}$

Subject to $\vec{w}A \leq \vec{c}$

$\vec{w}$ unrestricted

Example:

P: Minimize $6x_1 + 8x_2$

subject to

$$3x_1 + x_2 - x_3 = 4$$
$$5x_1 + 2x_2 - x_4 = 7$$
$$x_1, x_2, x_3, x_4 \geq 0$$

D: Maximize $4\omega_1 + 7\omega_2$

Subject to

$$3\omega_1 + 5\omega_2 \leq 6$$
$$\omega_1 + 2\omega_2 \leq 8$$
$$-\omega_1 \leq 0$$
$$-\omega_2 \leq 0$$

ω_1, ω_2 unristricted

$z^* = 8.4$	$z^* = 8.4$
$x_1^* = 1.4$	$w_1 = 0$
$x_3^* = 0.2$	$w_2 = 1.2$

Primal to dual conversion :

P : Minimize $\vec{c}\,\vec{x}$

Subject to $A\vec{x} \geq \vec{b}$

$\vec{x} \geq 0$

D : Maximize $\vec{w}\,\vec{b}$

Subject to $\vec{w}\,\vec{A} \leq \vec{c}$

$\vec{w} \geq 0$

Example :

P: Minimize $c_1 x_1 + c_2 x_2 + c_3 x_3$

Subject to
$$A_{11} x_1 + A_{12} x_2 + A_{13} x_3 \geq b_1$$
$$A_{21} x_1 + A_{22} x_2 + A_{23} x_3 \leq b_2$$
$$A_{31} x_1 + A_{32} x_2 + A_{33} x_3 = b_3 \quad \begin{matrix} \geq b_3 \\ \geq -b_3 \end{matrix}$$

$x_1 \geq 0,\ x_2 \leq 0,\ x_3$ unrestricted

Convert the problem into canonical form, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$

Minimize $c_1 x_1 - c_2 x_2' + c_3 x_3' - c_3 x_3''$

s.t.
$$A_{11} x_1 - A_{12} x_2' + A_{13} x_3' - A_{13} x_3'' \geq b_1 \quad w_1$$
$$-A_{21} x_1 + A_{22} x_2' - A_{23} x_3' + A_{23} x_3'' \geq -b_2 \quad w_2'$$
$$A_{31} x_1 - A_{32} x_2' + A_{33} x_3' - A_{33} x_3'' \geq b_3 \quad w_3'$$
$$-A_{31} x_1 + A_{32} x_2' - A_{33} x_3' + A_{33} x_3'' \geq -b_3 \quad w_3''$$

$$\text{Max} \quad b_1 w_1 - b_2 w_2' + b_3 w_3' - b_3 w_3''$$

$$
\begin{aligned}
w_1 A_{11} - w_2' A_{21} + w_3' A_{31} - w_3'' A_{31} &\le c_1 \\
-w_1 A_{12} + w_2' A_{22} - w_3' A_{32} + w_3'' A_{32} &\le -c_2 \\
w_1 A_{13} - w_2' A_{23} + w_3' A_{33} - w_3'' A_{33} &\le c_3 \\
-w_1 A_{13} + w_2' A_{23} - w_3' A_{33} + w_3'' A_{33} &\le -c_3
\end{aligned}
$$

$$w_1 \ge 0, \; w_2' \ge 0, \; w_3' \ge 0, \; w_3'' \ge 0$$

Example:

$$\text{Max} \quad 8x_1 + 3x_2 - 2x_3$$

$$
\begin{aligned}
\text{s.t.} \quad x_1 - 6x_2 + x_3 &\ge 2 \\
5x_1 + 7x_2 - 2x_3 &= -4 \rightarrow -5x_1 - 7x_2 + 2x_3 = 4
\end{aligned}
$$

$$x_1 \le 0, \; x_2 \ge 0, \; x_3 \text{ unrestricted}$$

Now $x_1 = -x_1', \; x_3 = x_3' - x_3''$

$$\text{Min} \quad -8x_1 - 3x_2' + 2x_3' - 2x_3''$$

$$
\begin{aligned}
\text{s.t.} \quad x_1 + 6x_2' + x_3' - x_3'' &\ge 2 \\
-5x_1 + 7x_2' + 2x_3' - 2x_3'' &\ge 4 \\
5x_1 - 7x_2' - 2x_3' + 2x_3'' &\ge -4
\end{aligned}
$$

$$
\begin{aligned}
D: \quad & \text{Max.} \quad 2w_1 + 4w_2' - 4w_2'' \\
& \text{s.t.} \quad w_1 - 5w_2' + 5w_2'' \leq -8 \\
& \qquad 6w_1 + 7w_2' - 7w_2'' \leq -3 \\
& \qquad w_1 + 2w_2' - 2w_2'' \leq 2 \\
& \qquad -w_1 - 2w_2' + 2w_2'' \leq -2
\end{aligned}
$$

$2w_1 + 4w_2$

The dual Simplex Method

This method solves dual problem directly on the (primal) simplex tableau.

The dual simplex method starts from optimal solution, but solution is infeasible.

At each iteration, feasibility is improved

a) Initialization step:

Find a basis B matrix such that

$$z_j - c_j = \vec{C}_B B^{-1} \vec{a}_j - c_j \leq 0 \text{ for all } j$$

(solution is optimal)

b) Main Step:

(i) If $\bar{b} = B^{-1} b \geq 0$, stop. Current solution is optimal.

Otherwise select pivot row with $\bar{b}_r < 0$;

$\bar{b} = \text{minimum}\,(\bar{b}_i)$. {$x_r$ is leaving variable}

(ii) If $y_{rj} \geq 0$ for all j; stop.

The dual is unbounded. Primal is infeasible.

Otherwise, perform minimum ratio

test over column k ;

$$\frac{z_k - c_k}{y_{rk}} = \underset{j}{\text{minimum}} \left\{ \frac{z_j - c_j}{y_{rj}} ; \ y_{rj} \leq 0 \right\}$$

(iii) perform row-column operations on pivot cell (r, k), and return to step b(i).

Dual Simplex Method

Min. $2x_1 + 3x_2 + 4x_3$

subject to

$$x_1 + 2x_2 + x_3 \geq 3$$

$$2x_1 - x_2 + 3x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

Multiply constraints to convert them into Dual form;

$$-x_1 - 2x_2 - x_3 \leq -3$$

$$-2x_1 + x_2 - 3x_3 \leq -4$$

Add variables x_4 and x_5;

$$-x_1 - 2x_2 - x_3 + x_4 = -3$$

$$-2x_1 + x_2 - 3x_3 + x_5 = -4$$

Transfer these values of objective function and constraints into Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	RHS	
	-2	-3	-4	0	0	0	Row-0
x_4	-1	-2	-1	1	0	-3	Row-1
x_5	-2	$+1$	-3	0	1	-4	Row-2

Note, the tableu contains optimal, but infeasible solution.

i) Select most $-$ve right-hand side row. Row-2 with x_5 as basic variable is selected. x_5 will leave solution

ii) Find minimum ratio for Row 0 and Row 1 entries for non-basic columns.

	x_1	x_2	x_3
Row-0	-2	-3	-4
Row-2	-2	$+1$	-3
	$\frac{-2}{-2}$	$\frac{-3}{+1}$ not allowed	$\frac{-4}{-3}$
Min Ratio	1	**	-1.33

x_1 enters the solution will cell (2,1) as pivot cell.

	$\downarrow$ x_1	x_2	x_3	x_4	x_5	RHS
	-2	-3	-4	0	0	0
x_4	-1	-2	-1	1	0	-3
$\leftarrow$ ~~x_5~~	-2	$+1$	-3	0	1	-4

Divide old-row 2 by -2

	x_1	x_2	x_3	x_4	x_5	RHS
	$2-2 = 0$	$-1-3=-4$	$-4+3=-1$	0	$0-1 = -1$	$\cancel{0}+4=4$
x_4	$\frac{-2}{-2}+1=0$	$-\frac{1}{2}+(-2) = -\frac{5}{2}$	$\frac{3}{2}-1=\frac{1}{2}$	1	$-\frac{1}{2}+0 = -\frac{1}{2}$	$\frac{-4}{-2}-3=-1$
x_1	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

- Divide old-row 2 by -2 and add to oldrow 1.
- ~~Divide~~ Multiply old-row 2 by -1 and add to old-row 0.

After calculations, new tableau has following entries;

	x_1	x_2	x_3	x_4	x_5	RHS
	0	-4	-1	0	-1	4
x_4	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	$\cancel{0}$ 1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

(i) If $\bar{b} = B^{-1}\vec{b} \geq 0$; stop [Current solution is optimal]

Otherwise select pivot row with $\bar{b}_r < 0$;

$\bar{b}_r$ = Minimum $\{\bar{b}_i\}$.

Since row-1 has $\bar{b}_1 < 0$; x_4 will leave solution.

(ii) If $y_{rj} \geq 0$; stop; [Solution is unbounded] {Primal is infeasible}.

Otherwise, find pivot column k as follows. from non-basic columns.

	x_2	x_5	x_3
row - 0	-4	-1	-1
row - 1	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
Ratio	$-4 \times \frac{-2}{5}$	-1×-2	*
	$= \frac{8}{5}$	$= 2$	*
	$= 1.6$	$= 2$	*

Min = {1.6}

Hence x_2 enters solution. Pivot Cell is (1,2)

	x_1	x_2	x_3	x_4	x_5	RHS
	0	-4	-1	0	-1	4
← x_4	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

pivot cell

Pivot operations for new table entries

a) Divide old-row 1 by $-\frac{5}{2}$ to get new-row 1

b) ~~Divide~~ Multiply old-row 1 by $-\frac{8}{5}$ and add to ~~old~~ old-row 0.

c) Multiply old-row ~~2~~ 1 by $-\frac{1}{5}$ and add to old-row 2.

x_1	x_2	x_3	x_4	x_5	RHS
0	$(-\frac{5}{2})(-\frac{8}{5})+4 = 0$	$(\frac{1}{2})(-\frac{8}{5})-1 = -\frac{9}{5}$	$(-\frac{8}{5}) = -\frac{8}{5}$	$(-\frac{1}{2})(-\frac{8}{5})-1 = -\frac{1}{5}$	$(-1)(-\frac{8}{5})+4 = \frac{28}{5}$
0	$\frac{-\frac{5}{2}}{-\frac{5}{2}} = 1$	$\frac{\frac{1}{2}}{-\frac{5}{2}} = -\frac{1}{5}$	$\frac{1}{-\frac{5}{2}} = -\frac{2}{5}$	$\frac{-\frac{1}{2}}{-\frac{5}{2}} = \frac{1}{5}$	$\frac{-1}{-\frac{5}{2}} = \frac{2}{5}$
1	$(-\frac{5}{2})(-\frac{1}{5})-\frac{1}{2} = 0$	$(\frac{1}{2})(-\frac{1}{5})+\frac{3}{2}$	$-\frac{1}{5}$	$(-\frac{1}{2})(-\frac{1}{5})-\frac{1}{2}$	$(-1)(-\frac{1}{5})+2$

x_3: $= -\frac{1}{10}+\frac{15}{10}$ $= \frac{14}{10}$ $= \frac{7}{5}$

x_5: $= \frac{1}{10}-\frac{1}{2}$ $= \frac{1}{10}-\frac{5}{10}$ $= -\frac{4}{10}$ $= -2/5$

RHS: $\frac{1}{5}+2$ $= \frac{1}{5}+\frac{2}{[illegible]}$ $= \frac{11}{5}$

After pivot-cell operations;

New Tableau is as follows;

	x_1	x_2	x_3	x_4	x_5	RHS
	0	0	$-\frac{9}{5}$	$-\frac{8}{5}$	$-\frac{1}{5}$	$\frac{28}{5}$
x_2	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$

Since $\vec{b} \geq 0$, as well as $z_j - c_j \leq 0$ for all j.

(i) Optimal Primal solution

$$x_1^* = \frac{11}{5}, \quad x_2^* = \frac{2}{5}$$

$$x_3 = x_4 = x_5 \equiv 0$$

(ii) Optimal dual Solutions

$$w_1^* = +\frac{8}{5}, \quad w_2^* = \frac{1}{5}$$

Economic Interpretation of Dual

P. Minimize $\vec{c}\,\vec{x}$
subject to $A\vec{x} \geq \vec{b}$
$\vec{x} \geq 0$

D: Maximize $\vec{w}\,\vec{b}$
Subject to $\vec{w}A \leq \vec{c}$
$\vec{w} \geq 0$

Let B = optimal basis for primal problem with $\vec{c}_B$ as its associated cost.
If $\vec{x}^*_B$ = optimal basic variables
Then

$$z = \vec{c}_B B^{-1} \vec{b} - \sum_{j \in R} (z_j - c_j) x_j$$
$$= \vec{w}^* \vec{b} - \sum_{j \in R} (z_j - c_j) x_j$$

For z^* to be optimal objective function;

$$z^* = \vec{w}^* \vec{b} \qquad \left(\because \text{ all } \vec{x_j} \equiv 0 \text{ since } \vec{x_j} \text{ are non-basic} \right)$$

$$\frac{\partial z^*}{\partial b_i} = \vec{c}_B B^{-1} = w_i^*$$

Hence w_i^* is a measure of rate of change of objective function value z^* w.r.t. change in value of b_i.

Example:

$$\text{Min} \quad -x_1 - 2x_2 + x_3 - x_4 - 4x_5 + 2x_6$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 6 \quad \text{(c.1)}$$

$$2x_1 - x_2 - 2x_3 + x_4 \leq 4 \quad \text{(c.2)}$$

$$x_3 + x_4 + 2x_5 + x_6 \leq 4 \quad \text{(c.3)}$$

$$\text{All } x_j \geq 0; \quad j = 1,2,3,4,5,6$$

Optimal Solution:

	Basis Inverse			RHS
	-2	0	-1	-16
x_2	1	0	$-\frac{1}{2}$	4
x_8	1	1	$-\frac{1}{2}$	8
x_5	0	0	$\frac{1}{2}$	2

$$z^* = (w_1, w_2, w_3)\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\frac{\partial z^*}{\partial b_1} = (c_2, c_8, c_5)\overset{B_1^{-1}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = (-2, 0, -1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -2$$

$$\frac{\partial z^*}{\partial b_2} = (c_2, c_8, c_5)\overset{B_2^{-1}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = (-2, 0, -1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\frac{\partial z^*}{\partial b_3} = (-2, 0, -1)\overset{B_3^{-1}}{\begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}} = -1$$

From Example problem; the optimal solution is; $x_2^* = 4$, $x_8^* = 8$, $x_5^* = 2$.

Constraint (C.1) is binding; since

x_1	x_2	x_3	x_4	x_5	x_6		RHS
0	4	0	0	2	0	= 6	6

Similarly, constraint (C.3) is binding, since

x_1	x_2	x_3	x_4	x_5	x_6		RHS
0	~~0~~	0	0	2*2	0	=4	4

$\frac{\partial z^*}{\partial b_1} = w_1^* = -2$. This means that objective will decrease (improve) by two units, if amount of b_1 is increased by one unit.

$\frac{\partial z^*}{\partial b_2} = w_2^* = 0$. There is no impact of b_2 on z^*.

$\frac{\partial z^*}{\partial b_3} = w_3^* = -1$. This means that objective function will decrease (improve) by one unit, if amount of b_3 is increased by one unit.